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Sample Solution

Question	1	2	3	4	5	6	7	8	9	10	11	Total
Max	5	12	5	12	10	8	8	8	16	6	10	100
CLO*	2	2	2	2	2	3	3	3	3	3	3	
Earned												

* Course Learning outcomes **Section 03**

Question 1: [5 Points] Induction [CLO 2]

Which statement is true? Circle the most suitable answer.

(a) Mathematical induction is based on the rule of inference that tells us that if $P(1)$ and $\forall k(P(k) \rightarrow P(k+1))$ are true for the domain of positive integers, then $\forall n P(n)$ is true.
(b) A proof by mathematical induction has two parts, a **basis step**, where we show that $P(1)$ is true, and an **inductive step**, where we show that for all positive integers k, if $P(k)$ is true, then $P(k+1)$ is true.
(c) In general, mathematical induction can be used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function.
(d) All the other statements are true.

Question 2: [12 Points] Induction [CLO 2]

Prove that $\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{(n+1)}$ For all positive integers n.

Basis step:
$n = 1$. Here both sides of the equation are equal to 1/2, so the claim holds.
Inductive step
We assume that $P(n) = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{(n+1)}$ is true for $n = k$. That is

$P(k) = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{(k+1)}$ is true. We need to show that $P(k+1)$ is true. That is

$P(k+1) = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{(k+2)}$

$P(k+1) = \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)(k+1)}{(k+1)(k+2)} = \frac{(k+1)}{(k+2)}$

This completes the proof.

Question 3: [5 Points] Induction [CLO 2]

Which statement is true? Circle the most suitable answer.

(a) $1 + 2 + \ldots + n = n(n + 1) / 2$.
(b) $n < 2^n$ for all integers $n \geq 1$.
(c) $2^n < n!$ for $n \geq 4$.
(d) All the other statements are true.

Question 4: [12 Points] Strong Induction [CLO 2]

Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. This question outline a strong induction proof that $P(n)$ is true for $n \geq 8$. Answer the following:

(a) Show that the statements $P(8)$, $P(9)$, and $P(10)$ are true, completing the basis step of the proof. [2 points]

$8 = 3 + 5$, $9 = 3 + 3 + 3$, $10 = 5 + 5$, so that $P(8)$, $P(9)$, and $P(10)$ are true.

(b) What is the inductive hypothesis of the proof? [3 points]

The inductive hypothesis is that $P(n)$ is true for $8 \leq n \leq k$, where $k \geq 10$. (Notice that this is a strong induction proof, which requires a stronger hypothesis.)

(c) What do you need to prove in the inductive step? [2 points]

We need to prove in the inductive step that $P(k + 1)$ is true.

(d) Complete the inductive step for $k \geq 10$. [3 points]

If $k \geq 10$, then $k + 1 = (k - 2) + 3$. Since $k - 2 \geq 8$, by the induction hypothesis we have that $P(k - 2)$ is true, i.e., a postage of 2 $k - 2$ cents can be paid by using 3-cent and 5-cent stamps. Adding one 3-cent stamp, we can pay a postage of $k + 1$ cents, i.e., $P(k + 1)$ is true.

(e) Explain why these steps show that this statement is true whenever $n \geq 8$. [2 points]

We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 8.

Question 5: [10 Points] Structural Induction and Recursion [CLO 2]

(a) define the height $h(T)$ of a full binary tree T recursively.

We define the height $h(T)$ of a full binary tree T recursively.

BASIS STEP: The height of the full binary tree T consisting of only a root r is $h(T) = 0$.

RECURSIVE STEP: If T_1 and T_2 are full binary trees, then the full binary tree $T = T_1 \cdot T_2$ has height $h(T) = 1 + \max(h(T_1), h(T_2))$.

(b) define the number of vertices $n(T)$ of a full binary tree T recursively.

We define the number of vertices $n(T)$ of a full binary tree T recursively as follows.

BASIS STEP: The number of vertices $n(T)$ of the full binary tree T consisting of only a root r is $n(T) = 1$.

RECURSIVE STEP*:* If T_1 and T_2 are full binary trees, then the number of vertices of the full binary tree $T = T_1 \cdot T_2$ is $n(T) = 1 + n(T_1) + n(T_2)$.

Question 6: [8 Points] The Pigeonhole Principle [CLO 3]

There are N students in a room; each student picks a day of the year to get a free dinner at a fancy restaurant. N is such that there must be at least one group of five students who select the same day. What is the smallest such N if the year is a leap year (366 days)? Show your steps to the solution.

The worst case is to have 366 students select 366 different days repeatedly. With this assumption we need N to be (4) . (366) + 1 = 1465 to guarantee that there must be at least one group of five students who select the same day.

Question 7: [8 Points] Counting and the Pigeonhole Principle [CLO 3]

Let S = {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21}. What is the smallest integer $N > 0$ such that for any set of N integers, chosen from S, there must be two distinct integers that one of them divides the other? Show your steps to the solution.

We have 1 divides all other numbers of the set:

(1,3), (1,5), (1,7) , (1,9) , (1,11) , (1,13) , (1,15) , (1,17) , (1,19) , (1,21)

We have 3 divides three numbers of the set: (3, 9), (3, 15), (3,21)

We have the following pairs one of them divides the other: (5, 15), (7, 21)

Hence, the following numbers do not divide other numbers in the list and no other number from the list divides them other than the number 1: 11, 13, 17, 19 (4 numbers). We choose these numbers and we can add to them 9, 15, 21. So the largest list that does not have two distinct integer that one divides the other is {9, 11, 13, 15, 17, 19, 21} (7 elements). A second list is {3, 5, 7, 11, 13, 17, 19} (7 elements). Adding any other number to the list will make the list has two distinct integers that one divides the other. So N = 8.

Question 8: [8 Points] Permutations and Combinations [CLO 3]

The owner of a pizzeria prepares every pizza by always combining 4 different ingredients. How many ingredients does he need, at least, if he would like to offer 35 different pizzas in the menu? Show your steps to the solution.

Using 6 different ingredients, he can obtain $\binom{6}{4} = 15$ different pizzas, while with 7 ingredients he gets $\binom{7}{4} = 35$ different pizzas. Then, since $\binom{6}{4} < 35 \leq \binom{7}{4}$, he needs at least 7 ingredients.

Question 9: [16 Points] Permutations and Combinations [CLO 3]

(a) How many strings of four decimal digits end with an even digit?

$10 \times 10 \times 10 \times 5 = 5000$

(b) How many strings of four decimal digits do not contain the same digit twice?

Assuming 0 is included: $10 \times 9 \times 8 \times 7 = 5040$
Assuming 0 is not included: $9 \times 8 \times 7 \times 6 = 3024$

(c) How many strings of four decimal digits have exactly three digits that are 9s?

$9 + 9 + 9 + 9 = 36$

The string can have: The non-9 as the first digit, OR the non-9 as the second digit, OR the non-9 as the third digit, OR the non-9 as the fourth digit. Thus, using the sum rule:

For each of those cases, there are 9 possibilities for the non-9 digit (any number other than 9) – Thus, the answer is $9 + 9 + 9 + 9 = 36$

(d) How many permutations of the letters ABCDEFG contain the string "CD"?

Count the permutation of AB$EFG where $ represents CD:

$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

Question 10: [6 Points] Binomial Coefficients and Identities [CLO 3]

The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}$, $0 \le k \le 10$, is:

1 10 45 120 210 252 210 120 45 10 1

Use Pascal's identity to produce the row immediately **before** this row in Pascal's triangle.

1 9 36 84 126 126 84 36 9 1

Question 11: [10 Points] Binomial Coefficients and Identities [CLO 3]

What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$? You may need to use the binomial formula $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}\,b^k$. Show your answer using combination and multiplication of numbers (no need to calculate a final answer).

Solution: Using the binomial formula taking a=2x, b=-3y, n=200 and k=99 we have that $x^{101}y^{99}$ term is $\binom{200}{99}(2x)^{101}(-3y)^{99} = -\binom{200}{99}2^{101}3^{99}x^{101}y^{99}$

Hence the desired coefficient is $-\binom{200}{99}2^{101}3^{99}$